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Solar system constraints on multifield theories of modified dynamics

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ABSTRACT

Any viable theory of modified Newtonian dynamics (MOND) as modified gravity is likely to require fields in addition to the usual tensor field of General Relativity. For these theories, the MOND phenomenology emerges as an effective fifth force probably associated with a scalar field. Here, I consider the constraints imposed on such theories by Solar system phenomenology, primarily by the absence of significant deviations from inverse-square attraction in the inner Solar system as well as detectable local preferred frame effects. The current examples of multifield theories can be constructed to satisfy these constraints and such theories lead inevitably to an anomalous non-inverse-square force in the outer Solar system.

Key words: gravitation – relativity – Solar system: general – dark matter.

1 INTRODUCTION

In modified Newtonian dynamics (MOND), it is postulated that the effective gravitational acceleration, g, deviates from the Newtonian value, $g_{\rm N}$, below a critical acceleration, a_0, in the sense that $g \approx \sqrt{a_0 g_{\rm N}}$ (Milgrom 1983). That the empirically determined value of a_0 ($\approx 10^{-8}$ cm s^{-2}) coincides with cH_0 to within an order of magnitude was immediately noted by Milgrom who speculated that MOND may reflect the influence of cosmology on local particle dynamics. In the context of General Relativity (GR), there is no such cosmological influence. This is essentially due to the fact that GR embodies the equivalence principle in its strong form which forbids environmental influence on local dynamics, apart from tides. However, if the gravitational force is partially mediated by a long-range scalar field as, for example, in Brans–Dicke theory, it is no longer the case that a local system is immune from cosmological influence. The scalar field, determined by the universal mass distribution and its time evolution, pervades the Universe and influences the gravitational dynamics of every subsystem. In Brans–Dicke theory, this influence is evidenced by the cosmic evolution of the effective gravitational constant (Brans & Dicke 1961).

This suggests that MOND may have its basis in scalar–tensor theory; indeed, the first relativistic theories proposed for MOND were scalar–tensor theories with non-standard aspects: the aquadratic Lagrangian theory, AQUAL (Bekenstein & Milgrom 1984), in which the scalar field Lagrangian is a general function of the usual scalar field invariant [$F(\phi^{,\alpha}\phi_{,\alpha})$], and phase-coupling gravity, or PCG, (Bekenstein 1988) in which the scalar field is complex with standard Lagrangian but only phase coupling to matter. Both these early attempts contain pathologies – superluminal propagation or instability of the background (Bekenstein & Milgrom 1984; Bekenstein 1990). Moreover, in these theories the scalar field is assumed to couple to matter jointly with the gravitational, or Einstein, metric in order to preserve the universality of free fall (Weak Equivalence Principle or WEP). But if that coupling is conformal, as in Brans–Dicke theory, then there is no enhanced gravitational deflection of photons due to the scalar field. This is in dramatic conflict with observations of lensing by clusters of galaxies (Bekenstein & Sanders 1994).

The lensing contradiction led to the idea that the relation between the Einstein and physical metrics should be more complicated than conformal; that is, the so-called 'disformal transformation' in which certain directions are picked out for additional dilation or contraction (Bekenstein 1993; Bekenstein & Sanders 1994). An initial proposal for such a theory (Sanders 1997) invoked a non-dynamical vector field, with only a time component in the preferred cosmological frame, to provide this additional stretching. The disformal coupling was combined with an aquadratic Lagrangian for the scalar field to yield the MOND phenomenology.

However, the non-dynamical aspect of the vector field violates general covariance, making it impossible to define a conserved energy–momentum tensor (Lee, Lightman & Ni 1974). This problem led Bekenstein (2004) to construct a tensor–vector–scalar theory (TeVeS) with a fully dynamical vector field; this theory, while yielding MOND phenomenology in the weak-field limit, is fully covariant, produces lensing at the same level as GR with dark matter, and possesses no obvious anomalies (propagation of scalar waves is causal). In the same vein, I proposed a biscalar tensor–vector theory (BSTV) in order to provide a cosmological origin of a_o and cosmological dark matter in the form of scalar field oscillations with wavelength sufficiently long to prevent clustering on the scale of galaxies (Sanders 2005).

Thus, it appears that any viable theory of MOND as modified gravity will require fields in addition to the tensor field of GR – a scalar field to yield the MOND phenomenology (as a fifth force) and a vector field to facilitate the non-conformal coupling and adequate gravitational lensing. Indeed, Soussa & Woodard (2004) have

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provided an elegant no-go argument to the effect that no single metric-based theory yielding MOND phenomenology in the weak-field limit can produce the necessary degree of gravitational lensing. The only other possibility for MOND as modified gravity is, then, a multifield theory such as TeVeS.

The purpose of this paper is to consider the constraints imposed on multifield theories of modified dynamics by Solar system phenomenology. The precession of the orbits of Mercury and Icarus, as well as limits on the variation of Kepler's constant, $G\ \mathrm{M}_\odot$, between the Earth and outer planets implies that the total force law within the orbit of Neptune is inverse square to high precision, apart from those post-Newtonian corrections introduced by GR. This suggests that any fifth force in the inner Solar system, in addition to preserving the WEP, is also precisely inverse square. Moreover, the absence of detectable post-Newtonian effects due to a scalar field tied to a cosmic rest frame, that is, ether-drift effects, probably constrains the magnitude of the fifth force to be less than 10^{-4} that of the normal gravity force. But, in the context of MOND, the anomalous force in the Galaxy, at the neighbourhood of the Sun, would have to be comparable to the gravity force. The transition from a weak inverse-square attraction in the inner Solar system to a significant anomalous attraction at several thousand astronomical units (au) would seem to require the appearance of a non-inverse-square acceleration in the outer Solar system.

This is interesting in view of the fact that a deviation from inverse-square attraction beyond 20 au is suggested based on Doppler data from the two Pioneer spacecrafts, the Pioneer anomaly (Anderson et al. 1998, 2002). The magnitude of this apparently constant anomalous acceleration ($\approx 8 \times 10^{-8}$ cm s^{-2}) is tantalizingly close to, although significantly larger than, the MOND acceleration (Turyshev, Nieto & Anderson 2006). I show here that an anomalous acceleration is an expected, and indeed predicted, aspect of multifield theories of modified dynamics. This non-inverse-square acceleration appears in the outer Solar system and need not be, but can be, as large as the observed Pioneer acceleration. Although the discussion is general, I illustrate this by considering current examples of multifield theories of MOND.

TeVeS, with Bekenstein's initial trial-free function, predicts a deviation which is too large to be consistent with both the reported constraints on $\Delta(G\ \mathrm{M}_\odot)$ and the probable limits on preferred frame effects. These contradictions are not fatal because the free function of the theory can be modified to produce an anomalous force consistent with the planetary and preferred frame constraints. Indeed the form of the free function required is also consistent with that demanded by observations of extended galaxy rotation curves which are flat beyond the visible disc. In this case the predicted anomalous acceleration appears beyond 100 au and is roughly $a_0/3$.

The BSTV is a modification of TeVeS constructed, in part, to be consistent with the constraints on deviations from inverse-square attraction and with the non-detection of preferred frame effects near the Earth. It also predicts a constant anomalous acceleration beyond Uranus that depends on the value of the scalar coupling strength. For values of the scalar field coupling constant below a critical value, the constant acceleration is also $\approx a_0/3$ as in TeVeS, but for larger couplings, the constant acceleration can be significantly larger than a_0 and extends within the orbit of Neptune; that is, the theory may be tuned to be consistent with the Pioneer effect. If so, however, it is then inconsistent with the reported limits on deviations from $1/r^2$ attraction out to the orbit of Neptune. This is unavoidable because if the Pioneer effect is really present within 30 au, then it would be inconsistent with limits on variation of $G\ \mathrm{M}_\odot$ between the orbits of the inner planets and the orbits of Uranus and Neptune – limits derived from spacecraft ranging to these two outer planets. Either this constraint on deviations from $1/r^2$ in the outer Solar system is too stringent, which is possible (Section 4.3), or the reported Pioneer anomaly has a standard explanation (not involving fundamental physics). A more radical possibility is that the Pioneer effect, and hence MOND, is not due to a modification of gravity but of the particle action (Milgrom 1994).

2 MULTIFIELD THEORIES OF MODIFIED DYNAMICS

2.1 General properties of multifield theories

I have emphasized that, in scalar–tensor theories of MOND, the relation between the physical and gravitational metrics cannot be conformal. This condition requires the introduction of a vector field, A^ν, that points in the time direction in the preferred cosmological frame. If the physical metric $\tilde{g}_{\mu\nu}$ is related to the gravitational metric $g_{\mu\nu}$ as

$$\tilde{g}_{\mu\nu} = \mathrm{e}^{-2\eta\phi} g_{\mu\nu} - 2\sin h(2\eta\phi) A_\mu A_\nu, \tag{1}$$

then it may be shown that the scalar field enhances the deflection of photons about a visible astronomical system exactly as it would be by appropriately added dark matter in the context of pure GR; that is, relativistic and non-relativistic particles feel the same total weak-field force. (Sanders 1997; Bekenstein 2004). Here η is a parameter describing the strength of the scalar coupling to matter and is related to the parameter k in Bekenstein's notation ($\eta^2 = k/4$).

It is useful to discuss scalar–tensor theories of modified dynamics in the context of the Einstein frame where the scalar field ϕ may be considered to mediate a force, $\boldsymbol{f}_\mathrm{s}$, in addition to the usual gravity force connected to the gravitational tensor, the Einstein–Newton force $\boldsymbol{f}_\mathrm{N}$; that is to say, in this frame, particle motion is generally non-geodesic. In such theories, the phenomenology associated with MOND results from this 'fifth force' which is, in the extragalactic domain, a non-inverse-square force that dominates in the regime of low field gradients ($f_\mathrm{s} = \eta c^2 \nabla\phi < a_0$).

The aspect of non-inverse-square attraction requires a departure from the standard Lagrangian for scalar–tensor theories ($L_\mathrm{s} = \phi^{,\alpha}\phi_{,\alpha}$) in the form of either the aquadratic theory with a non-standard scalar Lagrangian ($F(L_\mathrm{s})$) or a biscalar theory where one field couples to matter and the second determines the strength of that coupling (as in PCG). Each of these prescriptions may be designed to provide a scalar force about a point mass of the form

$$f_\mathrm{s} = \eta c^2 \nabla\phi = \frac{\sqrt{GMa_0}}{r}, \tag{2}$$

at least in the regime where $f_\mathrm{s} < a_0$. The total weak-field force would then be given by $f_\mathrm{t} = f_\mathrm{s} + f_\mathrm{N}$ where $f_\mathrm{N} = GM/r^2$ is the usual Newtonian force; clearly f_s given by equation (2) will dominate at accelerations below a_0.

TeVeS is an aquadratic theory in disguise, with a scalar field action that may be written as

$$L_\mathrm{s} = \mu\eta^2 \boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla}\phi + V(\mu). \tag{3}$$

This is the weak coupling limit of PCG ($\eta \ll 1$), the AQUAL limit, where one may show that the kinetic term for μ vanishes. As written here, μ is an auxiliary non-dynamical field and is algebraically related to $(\nabla\phi)^2$ via the potential function $V(\mu)$. This relation, expressed in terms of the scalar force $f_\mathrm{s} = \eta\nabla\phi c^2$ is given by

$$\left(\frac{f_\mathrm{s}}{a_0}\right)^2 = -{l_\mathrm{M}}^2 V'(\mu), \tag{4}$$

where $V' = \mathrm{d}V/\mathrm{d}\mu$ and $l_{\rm M} = c^2/a_0$. Here, I will refer to $-l_{\rm M}\,V'(\mu)$ as the free function of the theory, although this function has, as its basis in the Lagrangian, the potential of a possibly dynamical field, $V(\mu)$. In the weak-field static limit, equation (3) leads to the well-known Bekenstein–Milgrom field equation

$$\nabla\,[\mu(|f_{\rm s}|/a_0)\boldsymbol{f}_{\rm s}] = 4\pi G\rho. \tag{5}$$

The function $\mu(x)$ as it appears in equation (5) does not have the same meaning as $\tilde{\mu}$ in the original MOND prescription $[f_{\rm t}\tilde{\mu}(|f_{\rm t}|/a_0) = f_{\rm N}]$ or in the single-field Bekenstein–Milgrom non-relativistic theory (Bekenstein 2004). Equation (5) applies only to the scalar component of the force. In the context of such multifield theories, not all forms of $\tilde{\mu}$ are realizable from sensible single-valued forms of μ (Zhao & Famaey 2006).

The most obvious, and simplest, choice for the free function would be

$$\left(\frac{f_{\rm s}}{a_0}\right)^2 = \mu^2, \tag{6}$$

or $\mu(x) = x$ for all x. This corresponds to $V(\mu) = -\mu^3/(3l_{\rm M}^2)$ and leads to a scalar force of the form of equation (2) at all r; of course, the Newtonian force dominates for $f_{\rm t} > a_0$. The rotation curves of spiral galaxies would be *asymptotically* flat as in MOND and would satisfy a mass–rotation velocity relation (Tully–Fisher) of the form $v^4 \propto M$. We see below, however, that such a theory is inconsistent with the observed form of galaxy rotation curves as well as tight constraints on deviations from inverse-square attraction in the inner Solar system.

2.2 Rotation curve constraints on fifth force theories

A more complicated scalar field Lagrangian is provided by the Bekenstein free function

$$\left(\frac{f_{\rm s}}{a_0}\right)^2 = \frac{1}{4}\mu^2(\eta^2\mu - 2)^2(1 - \eta^2\mu)^{-1} \tag{7}$$

[here μ as defined by equation (5) differs by a factor of η^2 from Bekenstein's definition]. This yields a scalar force illustrated by the dotted curve in Fig. 1 where we see a return to $1/r^2$ attraction at high accelerations (here $\eta = 0.01$). For $f_{\rm s}/a_0 < 10^{-4}$, this is equivalent to scalar force dependence provided by equation (6).

This free function, as well as that described by equation (6), is unacceptable in that the form of the observed rotation curves of spiral galaxies implies that the scalar force cannot continue to increase smoothly as $1/r$ for accelerations near a_0; the resulting rotation curves decline too slowly to the asymptotically constant value. This has been demonstrated for the Milky Way galaxy and for the well-studied spiral galaxy, NGC 3198 (Famaey & Binney 2005), and it is generally true (Zhao & Famaey 2006). Fig. 2 shows the Newtonian rotation curve (solid curve) resulting from a spherically symmetric mass distribution of galaxy scale mass ($10^{11}\,{\rm M}_\odot$), an exponential sphere with a length-scale of 2 kpc. The dotted curve is the rotation curve resulting from TeVeS with the free function described by equation (7). The slow decline to the asymptotic value is evident.

A free function given by

$$\left(\frac{f_{\rm s}}{a_0}\right)^2 = (\mu^2 + 2\mu^4)(1 + 4\mu^2)^{-2}[1 - 2ln(1 - \eta^2\mu^2)] \tag{8}$$

would give rise to a radial dependence for the scalar force about a point mass of the form shown by the long-dashed curve in Fig. 1. This is qualitatively similar to that originally suggested for the aquadratic

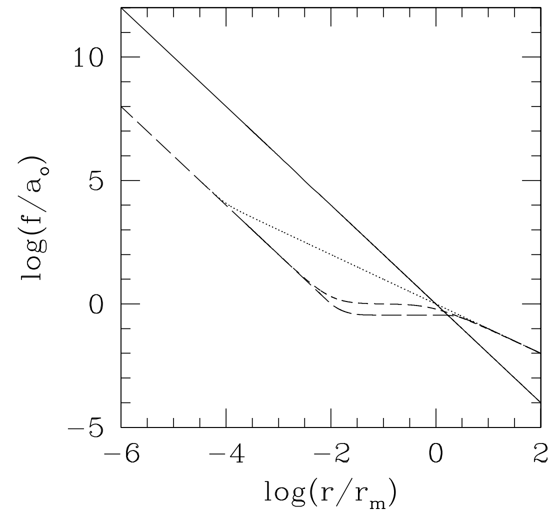

Figure 1. The log of the Newtonian force, $f_{\rm N}$ and scalar force, $f_{\rm s}$, in units of 10^{-8} cm s^{-2} plotted against the log of the radial distance from a point mass in units of the MOND radius ($r_{\rm m} = \sqrt{GM/a_0}$). The solid curve is the Newtonian force. The dotted curve is the scalar force for TeVeS with the free function originally taken by Bekenstein (2005). The short-dashed curve is the scalar force resulting from the free function suggested by Zhao & Famaey (2006) and the long-dashed curve is the scalar force corresponding to equation (8) here. In all cases, $\eta = 0.01$.

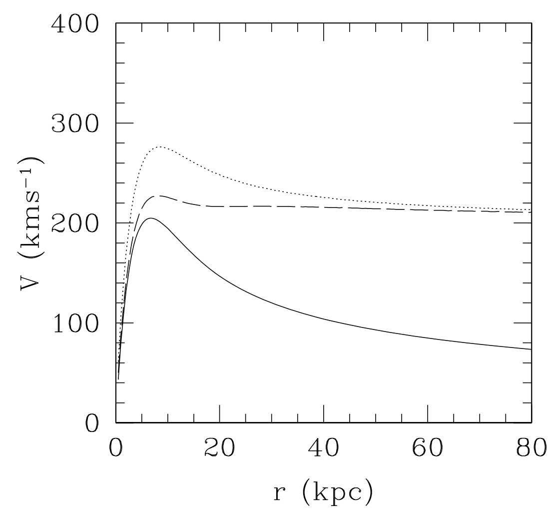

Figure 2. Rotation curves for a spherical galaxy resulting from TeVeS with two alternative forms of the free function. The density distribution is taken to be exponential with a scalelength of 2 kpc and the total mass is $10^{11}\,{\rm M}_\odot$. The dashed line corresponds to the original Bekenstein free function (equation 7) and the dotted curve to that of the free function considered here (equation 8). The solid curve is the Newtonian curve.

stratified theory (Sanders 1997) and to the radial force dependence provided by the free function proposed by Zhao & Famaey (2006) (short-dashed curve). Here, we see that for total accelerations greater than a_0 the scalar force becomes constant, $f_{\rm s} \approx a_{\rm p} \approx a_0/3$ before resuming $1/r^2$ dependence at larger total accelerations. The MOND interpolating function, $\tilde{\mu}$ (for spherical symmetry) corresponding to the free functions of Bekenstein, Zhao–Famaey, and equation (8) are shown in Fig. 3. Consistency with observed galaxy rotation curves requires a transition to the Newtonian regime at least as rapid as that provided by the Zhao–Famaey free function.

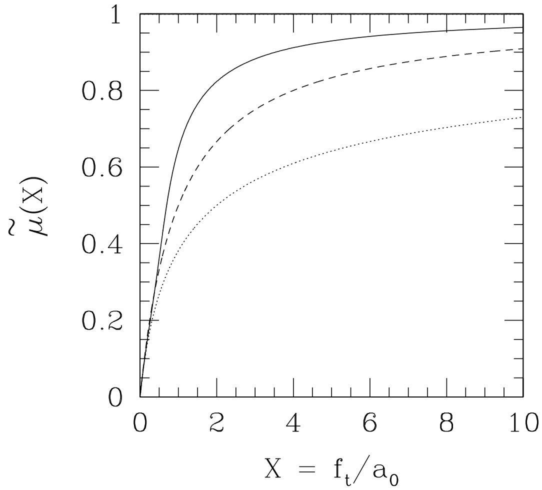

Figure 3. The interpolating function $\tilde{\mu}$ in the MOND prescription ($f_t\tilde{\mu}(|f_t|/a_0) = f_N$) corresponding to the Bekenstein free function (equation 7, dotted curve), the Zhao–Famaey free function (dashed curve), and the free function considered here (equation 8, solid curve). The indicated form is only strictly valid for spherical symmetry, and in all cases $\eta = 0.01$ As emphasized by Zhao & Famaey (2006) with the Bekenstein free function the transition between the MOND and Newtonian regimes is too gradual to yield agreement with observed rotation curves.

The relative merit of various free functions is not the topic here; the point is that radial dependence of scalar force must become rather flat at accelerations larger than a_0 to be consistent with galaxy rotation curves. This is evident in Fig. 2 where the rotation curve resulting from TeVeS with the free function described by equation (8) is shown by the long-dashed curve; this is more consistent with observed rotation curves which are generally flat beyond the visible disc. As will be shown below, this same qualitative behaviour is also required to meet the constraints on deviations from inverse-square attraction in the inner Solar system while avoiding observable preferred frame effects.

The BSTV theory has been designed to produce a radial dependence of the scalar force similar in form to that given by TeVeS with the modified free function, equation (8), (dashed curve in Fig. 1), and therefore, rotation curves of the observed form (see figs 1 and 2 in Sanders 2005). There is, however, one important difference: because the field determining the strength of the scalar coupling in this theory, equivalent to μ, is dynamical (unlike TeVeS where it is an auxiliary field only), the value of the constant scalar acceleration near $f_s = a_0$ depends not only on the scalar coupling strength, η, but also on the value of the source mass and its distribution. This can make an important difference in outer Solar system phenomenology.

3 SOLAR SYSTEM CONSTRAINTS

3.1 Planetary motion

The most sensitive natural gravity probe in the inner Solar system is provided by the orbit of Mercury. As is well known, General Relativity found its first experimental success in providing a non-Newtonian explanation of the anomalous precession of this orbit. The precise prediction is $\Delta\theta = 43.03$ arcsec per century, and the present observational result agrees with this to better than 0.05 arcsec per century (Will 2001). That is to say, precession resulting from any additional non-Newtonian effect must be less than this limit. This provides a strong constraint on long-range fifth-force models.

Here, I will parametrize an additional non-Newtonian force in terms of a constant acceleration a_p. It is straightforward to demonstrate that the precession introduced by such a constant acceleration would be given by

$$\Omega_p = -a_p(1-e^2)^{1/2}\left(\frac{a}{G\,\mathrm{M}_\odot}\right)^{1/2}, \tag{9}$$

where e is the eccentricity of the orbit and a is the semimajor axis. This may be rewritten as

$$\Omega_p = 6.5(1-e^2)^{1/2}\left(\frac{a_p}{10^{-8}\ \mathrm{cm\,s^{-2}}}\right)\left(\frac{10\ \mathrm{km\ s^{-1}}}{V}\right)\ \mathrm{arcsec/century}, \tag{10}$$

where V is the mean orbital velocity. If, for Mercury, $\Omega_p < 0.05$ arcsec/century, equation (10) would imply that $a_p < 4.0 \times 10^{-10}$ cm s^{-2}. That is to say, any constant anomalous acceleration present at the distance of Mercury from the Sun, must be 200 times smaller than the reported Pioneer acceleration.

The precision of measured planetary precession degrades rapidly for the other terrestrial planets, but the asteroid, Icarus, remains a useful probe because of its near-Earth passage ($a = 1.08$ au) and its high eccentricity ($e = 0.83$). Here, the precession predicted from GR is 10.3 arcsec/century and the observed precession is $\Omega_p = 9.8 \pm 0.8$ arcsec/century (Weinberg 1972). Taking 0.8 arcsec/century as the limit on precession due to a constant acceleration, we find, from equation (10), that $a_p < 6.3 \times 10^{-8}$ cm s^{-2}.

Beyond Icarus, the tightest constraints on deviations from $1/r^2$ attraction are provided by limits on the variation of Kepler's constant, $K_p = G\ \mathrm{M}_\odot$. If a variation, ΔK_p is detected between two planetary orbits, and if this is parametrized by the presence of a constant acceleration, a_p, then

$$a_p = \frac{\Delta K_p}{K_p} K_p \left({r_2}^2 - {r_1}^2\right)^{-1}, \tag{11}$$

where r_1 and r_2 are the distances from the Sun of the closer and more distant planets, respectively (assumed to be on circular orbits). An additional scalar force described by equation (2) (in addition to the Newtonian force) would result in a variation of Kepler's constant given by

$$\frac{\Delta K_p}{K_p} = \frac{r_2 - r_1}{r_m} \tag{12}$$

($r_m = \sqrt{G\,\mathrm{M}_\odot/a_0} \approx 7700$ au).

Since the advent of interplanetary spacecrafts, the distances to the planets are known to high accuracy. This provides strict limits on the variation of Kepler's constant between the Earth and the planet in question, that is,

$$\frac{\Delta K_p}{K_p} = \frac{1}{2}\frac{\Delta P}{P} + \frac{3}{2}\frac{\Delta r}{r}, \tag{13}$$

where $\Delta P/P$ is the uncertainty in the period and $\Delta r/r$ is the uncertainty in the solar distance. For example, from the Viking mission to Mars, it is known that the uncertainty in the difference in the orbital radii of Earth and Mars is less than 100 m. Moreover, the difference in the orbital periods between the Earth and Mars is known to better than seven parts in 10^{11}. By equation (13), this implies that $\Delta K_p/K_p < 2 \times 10^{-9}$ and, hence, from equation (11), $a_p < 0.1 \times 10^{-8}$ for any constant acceleration present between the orbits of the Earth and Mars (Anderson et al. 2002).

Similar observational limits resulting from Pioneer and Voyager flybys constrain $\Delta K_p/K_p$ between the inner planets and Jupiter,

Table 1. Planetary constraints on a constant anomalous acceleration.

Object	Distance (au)	Method	$a_{\rm p}$ (10^{-8} cm s^{-2})
Mercury	0.39	$\Omega_{\rm p}$	0.04
Icarus	1.08	$\Omega_{\rm p}$	6.3
Mars	1.52	$\Delta K_{\rm p}/K_{\rm p}$	0.1
Jupiter	5.2	$\Delta K_{\rm p}/K_{\rm p}$	0.12
Uranus	19.2	$\Delta K_{\rm p}/K_{\rm p}$	0.08*
Neptune	30.1	$\Delta K_{\rm p}/K_{\rm p}$	0.13*

The final column is the upper limit on a constant anomalous acceleration determined from planetary orbits via the indicated method ($\Omega_{\rm p}$, planetary precession; $\Delta K_{\rm p}/K_{\rm p}$, variation of Kepler's constant). The constraints imposed by the orbits of Uranus and Neptune (marked with asterisks) are controversial (see the text), but if valid would be inconsistent with the Pioneer anomaly as a modification of gravity. The distance given is the semimajor axis of the orbit.

Uranus and Neptune to be less than .12, 0.5 and 2.0 × 10^{-6}, respectively (Anderson et al. 1995). The corresponding limits on a constant acceleration are .26, .08, and .13 in units of 10^{-8} cm s^{-2}. We see that these limits for Uranus and Neptune are inconsistent with the reported Pioneer anomaly if the anomaly is present at distances beyond 20 au. These results are summarized in Table 1.

There is disagreement over these outer Solar system constraints (Section 4.3), but in any case, it is clear that the $1/r$ dependence of a fifth force cannot continue into the inner Solar system – certainly not to within the orbit of Mars ($r/r_{\rm m} = 2 \times 10^{-4}$) – because here the total gravitational field is so nearly inverse square. A force law of the form of equation (2) would result in $\Delta K_{\rm p}/K_{\rm p} = 7.4 \times 10^{-5}$ (equation 12) or 30 000 times larger than the observed limit. This suggests that in TeVeS the radial dependence of scalar force must be quite precisely $1/r^2$ certainly within the orbit of Mars. As we see from Fig. 1 this is consistent with the free function given by equation (8) or by Zhao & Famaey – forms which are also consistent with observed galaxy rotation curves. Note that a theory can be constructed in which the net scalar force vanishes within a_0. This is highly contrived (involving two scalar components, one attractive and one repulsive), but implies that, in all that follows, one should add the condition that the scalar force is a monotonically decreasing function of radius.

3.2 Post-Newtonian constraints

Given that the total force must be quite precisely $1/r^2$ in the inner Solar system, it is reasonable to suppose that the scalar force, in the high-acceleration regime, is also $1/r^2$ as in Brans–Dicke theory. Then one may ask if there is any restriction on the ratio of the weak-field scalar to Newtonian forces in the inner Solar system, $f_{\rm s}/f_{\rm N}$. In Brans–Dicke theory, $f_{\rm s}/f_{\rm N} = \eta^2 = 1/(2\omega + 3)$ (ω is the Brans–Dicke measure of scalar coupling strength). Because both $f_{\rm s}$ and $f_{\rm N}$ are inverse square in the weak-field limit, there is no restriction on the ratio of forces, or ω, from weak-field phenomenology; the restrictions appear at the post-Newtonian level.

In isotropic coordinates, the metric about a point mass may be written as

$$\mathrm{d}\tau^2 = \left[1 - \frac{2GM}{rc^2} + 2\beta\left(\frac{GM}{rc^2}\right)^2 + \cdots\right]\mathrm{d}t^2 - \left(1 + 2\gamma\frac{GM}{rc^2}\right)\mathrm{d}r^2, \tag{14}$$

where the coefficients γ and β – the Eddington–Robertson parameters – describe the lowest order relativistic deviations from Newtonian inverse-square gravity (post-Newtonian). In GR $\gamma = \beta = 1$ precisely, and in Brans–Dicke theory, it may be shown that $\beta = 1$. In fact, this is true of any conformally coupled scalar–tensor theory, $\tilde{g}_{\mu\nu} = \psi(\phi)g_{\mu\nu}$, provided that

$$\psi''(0) = [\psi'(0)]^2 \tag{15}$$

(see Appendix A). But, as noted in Introduction, because of the conformal relation between the physical and gravitational metrics, there is no enhanced deflection of photons due to the scalar field, while non-relativistic particles do respond to an enhanced force. This is reflected in the fact that the post-Newtonian parameter $\gamma = (\omega + 1)/(\omega + 2) \neq 1$ in Brans–Dicke theory. In general, $\gamma \neq 1$ in in conformally coupled scalar–tensor theories.

A disformal transformation of the form of equation (1) has been discussed by Giannios (2005) in the context of TeVeS. It is equivalent to multiplying different components of $g_{\mu\nu}$ by separate functions of ϕ; that is, $\tilde{g}_{\rm tt} = \psi(\phi)g_{\rm tt}$ and $\tilde{g}_{rr} = \chi(\phi)g_{rr}$. In such theories, it is the case that $\gamma = 1$ if

$$\chi'(0) = -\psi'(0) \tag{16}$$

(see Appendix A). If both conditions (15) and (16) are met and if $A^r = 0$ (i.e. the vector does not develop a non-zero radial component), then $\beta = \gamma = 1$. It is easy to verify that the particular transformation provided by equation (1), where $\psi(\phi) = \exp(2\eta\phi)$ and $\chi(\phi) = \exp(-2\eta\phi)$, satisfies conditions (15) and (16). Therefore, for any TeVeS in which the gravitational and physical metrics are related according to equation (1), there is no restriction on $f_{\rm s}/f_{\rm N}$ at post-Newtonian level as described by the standard Eddington–Robertson parameters. This is true of the classical stratified theories (Ni 1972), of the stratified aquadratic theory (Sanders 1997) and of TeVeS, assuming $A^r = 0$. These theories are consistent with a wide range of observed phenomena from deflection of starlight by the Sun to radar echo delay. It is the gravitational preferred frame effects that are threatening for such theories.

3.3 Preferred frame constraints

Multifield theories of MOND must contain a normalized cosmic vector field to provide the disformal transformation. The direction of the vector is determined primarily by the universal mass distribution and, in Friedmann–Robertson–Walker (FRW) metric, points in the positive time direction. The equations of motion in a gravitational field take their simplest form in the cosmic frame where only the time component of the vector field is non-zero. For a frame in relative motion, such as the Solar system, space components of the vector field develop non-zero values, and this affects the motion of particles; that is, ether-drift effects must appear at some level; that is, such theories violate the Lorentz invariance of gravitational dynamics.

Post-Newtonian preferred frame effects in conservative theories are quantified by two parameters, α_1 and α_2 (Will & Nordtvedt 1972). These modify the effective Lagrangian that describes the gravitational dynamics of N-body systems by adding terms such as

$$\delta L_{\alpha_1} = -\frac{\alpha_1}{4}\sum_{\rm A\neq B}\frac{GM_{\rm A}M_{\rm B}}{r_{\rm AB}\,c^2}\boldsymbol{V}_{\rm A}\cdot\boldsymbol{V}_{\rm B} \tag{17}$$

and

$$\delta L_{\alpha_2} = \frac{\alpha_2}{4}\sum_{\rm A\neq B}\frac{GM_{\rm A}M_{\rm B}}{r_{\rm AB}\,c^2}(\boldsymbol{w}\cdot\hat{\boldsymbol{r}}_{\rm AB})^2, \tag{18}$$

where $\boldsymbol{V}_A$ is the velocity of particle A with respect to the preferred cosmic frame, $\boldsymbol{w}$ is the velocity of the inertial frame with respect to the cosmic frame, and $\hat{\boldsymbol{r}}_{AB}$ is the unit vector along $\boldsymbol{r}_{AB}$. A non-zero value of α_1 would lead to effects such as a polarization of the Earth–moon orbit and is constrained to be less than 10^{-4} by lunar laser ranging (LLR; Müller, Nordtvedt & Vokrouhlický 1996). The α_2 term quantifies effects such as periodic variation in the effective gravitational constant (with twice the orbital or rotational frequency of the system) or an ether-drift torque acting on a spinning body. This is constrained to be less than 10^{-7} by the near-alignment of the Sun's rotational axis with that of the Solar system (Nordtvedt 1987).

Calculation of the predicted values of α_1 and α_2 must be done for each particular theory. Here to keep the discussion as general as possible, I provide estimates of the preferred frame parameters in TeVeSs by heuristic arguments.

In the historical Lagrangian-based stratified theories such as that of Ni (1972), there is one dynamical field, ϕ, a non-dynamical tensor (Minkowski) describing the background geometry, and a non-dynamical vector field; that is to say, the gravitational force is supposed to be mediated only by the scalar field disformally coupled to an a priori geometry described by the Minkowski metric. Here, it may be shown that $\alpha_2 = 0$ but $\alpha_1 = -8$ in sharp contradiction with the LLR result, not to mention earlier constraints on the diurnal and annual variation of the gravitational constant.

In the predecessor to TeVeS, the aquadratic stratified theory (Sanders 1997), there are two dynamical fields, a scalar and the Einstein metric, in addition to a non-dynamic vector field. That is to say, the gravitational force is mediated not only by the scalar field, but also by the Einstein metric which is locally insensitive to motion through the cosmic frame. Here, the preferred frame effects would appear through the contribution of the scalar to the physical metric (equation 1). In the limit where the scalar coupling, η, vanishes, the theory reduces to GR and in GR there are no preferred frame effects. Therefore, observational limits on preferred frame effects must place an upper limit on η.

This can be made more definite by noting that the equation of motion for the scalar field, in the high acceleration limit where the Lagrangian is standard, has the form

$$[\phi^{\alpha;\beta}]_{;\beta} = 4\pi G\eta\tilde{T}_{\mu\nu}[g^{\mu\nu} + (1 + e^{-4\eta\phi})A^{\mu}A^{\nu}] \tag{19}$$

where $\tilde{T}_{\mu\nu}$ is the usual energy–momentum tensor in the physical frame (Bekenstein 2004). In a frame moving with velocity w with respect to the cosmic frame, the source, to order w^2/c^2 would take the form $4\pi G\eta\ \rho[1 + (w^2 + wv)/2c^2]$, but $g_{\mu\nu}$ would contain no such terms to post-Newtonian order. Since $f_s = \eta\nabla\phi$, I conjecture that α_1 is suppressed, relative to its value in pure stratified theories, by $\approx f_s/f_N$. If so, this would constrain $f_s/f_N < 10^{-4}$

Similar arguments would apply in a theory such as TeVeS where all three fields, including the vector, are dynamical; that is, we would expect post-Newtonian preferred frame effects to project into the Solar system via the scalar field which is tied to the cosmological frame. There is, however, an additional effect because the vector field contributes directly to the source of the Einstein tensor, $G_{\mu\nu}$. This contribution is not negligible because of the presence, in the theory, of a Lagrangian multiplier function included to enforce a normalization condition on the vector field, $A_{\mu}A^{\mu} = -1$. The additional term in the energy–momentum tensor is then $-\lambda A_{\mu}A_{\nu}$ which, from the vector field equation becomes $\approx 4\pi KG_N\rho A_{\mu}A_{\nu}$ where K is a new parameter associated with the vector field (a coupling strength parameter), and G_N is the locally measured gravitational constant [the primary effect of the vector field in the weak-field limit is a rescaling of the gravitational constant with respect to its cosmological value, G, that is, $G_N \approx G(1 - K/2)^{-1}$]. Therefore, given a Lorentz transformation of the cosmic vector field to a moving frame, we see that the source of $G_{\mu\nu}$ contains terms proportional to $KG_N\ \rho w^2/c^2$. This would constrain K to be less than α_1 or α_2 (say $K < 10^{-7}$) but would have no profound effect on weak-field phenomenology.

In summary then, we can say that the suppression of the likely preferred frame effects such as polarization of the Earth–moon orbit will probably require that $f_s/f_N < 10^{-4}$. This is comparable to the current reported constraints on a scalar force in the context of Brans–Dicke theory (Bertotti, Iess & Tortora 2003). But I re-emphasize, this is a heuristic argument and a proper calculation should be done.

4 CONFRONTATION OF MULTIFIELD THEORIES WITH SOLAR SYSTEM PHENOMENOLOGY

Summarizing the above discussion, we have seen, first of all, that the total weak-field gravitational force in the Solar system is inverse square to high precision, at least within the orbit of Mars. This implies that any component of the gravitational force, in addition to the Einstein–Newton force, should also be precisely inverse square. At the same time, post-Newtonian preferred frame effects would seem to require that any additional inverse-square force due to a scalar field should be smaller than 10^{-4} of the Einstein–Newton force. It is not trivial for a theory to satisfy these two constraints.

4.1 TeVeS

In TeVeS, the relevant parameter, which determines the strength of the scalar force, is η; that is, $f_s/f_N = \eta^2$ in the limit where $f_s \gg a_0$. The preferred frame considerations would then require the $\eta < 10^{-2}$. In Fig. 4, the dotted curve shows the anomalous force (the non-$1/r^2$ component of the total force) resulting from Bekenstein's initial free function (equation 7), with $\eta = 0.01$, compared to the

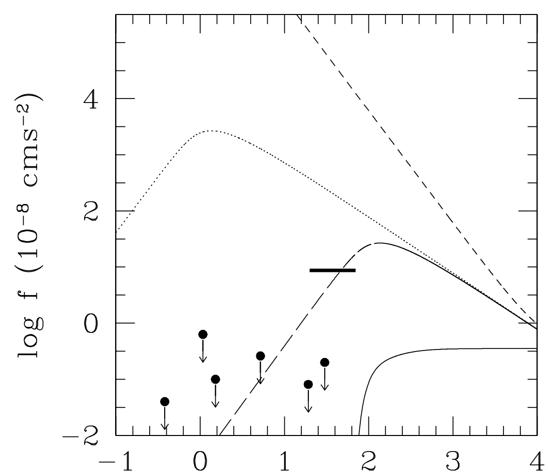

Figure 4. The dashed curve is the log of the total force ($f_t = f_s + f_N$), in units of 10^{-8} cm s^{-2} plotted against the log of the radial distance from the Sun in astronomical units for TeVeS. The dotted curve is the anomalous force (the non-inverse-square force) for Bekenstein's initial choice of free function with $\eta = 0.01$ (equation 7). The long-dashed curve is the same but with $\eta = 0.1$. The solid curve is the anomalous force resulting from the modified free function (equation 8). The points show the limits on a peculiar acceleration implied by planetary motion as discussed in Section 2, and the solid bar is the Pioneer acceleration.

total force within the Solar system (dashed curve). The points are the limits on a constant anomalous acceleration from planetary motion discussed in Section 2. The solid bar represents the Pioneer anomalous acceleration. Here, it is obvious that the theory, with this choice of free function and scalar coupling strength, strongly violates these limits on deviations from inverse-square attraction well into the inner Solar system.

The deviation from inverse-square attraction is less severe if the scalar coupling constant, η is larger. The anomalous force resulting when $\eta = 0.1$ is shown by the long dashed curve; this would appear to be roughly consistent with the constraints on deviations from inverse square within the inner Solar system. But then the scalar force is only 0.01 of the Newton–Einstein force, and the theory would probably evidence local preferred frame effects at least 100 times larger than the present limits.

The modified form of the free function (equation 8) with $\eta = 0.01$, gives an anomalous force shown by the solid curve in Fig. 4. It is obvious that this is consistent both with the planetary constraints on deviations from $1/r^2$ attraction out to Neptune and with the avoidance of local preferred frame effects required by $f_s/f_N < 10^{-4}$, but a constant anomalous acceleration $\approx a_0/3$ does appear beyond 100 au.

4.2 BSTV

BSTV is in part designed to satisfy these Solar system constraints but at the expense of adding a new parameter $\epsilon > \eta$ (the parameter is not necessary an additional; it may be identified with the vector coupling strength). Here, there are two explicitly dynamical scalar fields – one, ϕ, that couples to matter and the second q which determines the strength of the coupling. In terms of the scalar field gradient, the quasi-static field equation is

$$\nabla [q^2 \nabla\phi] = \frac{8\pi G\eta\rho}{c^2}, \tag{20}$$

where $q^2 \rightarrow \epsilon^2$ in the high acceleration limit; that is, q saturates at a small value in this limit (note that $\mu = q^2/2\eta^2$ with μ as defined in equations 3 and 5). Given that η is the strength of the scalar field coupling (as in equation 1) and that the scalar force is $f_s = \eta c^2 \nabla\phi$, the theory is designed to yield a precisely $1/r^2$ force in the inner Solar system with $f_s/f_N \rightarrow 2\eta^2/\epsilon^2$ in the limit where $f_s \gg a_0$. Thus, for this theory, the avoidance of preferred frame effects near the Earth would require that $2\eta^2/\epsilon^2 < 10^{-4}$

But there is another significant difference with TeVeS. The relation between the coupling strength field, q, and the scalar force is no longer algebraic (as in equations 7 and 8) but is differential and given by

$$\nabla^2 q - q \boldsymbol{\nabla}\phi\, \boldsymbol{\nabla}\phi = V_s'(q), \tag{21}$$

where $V_s(q)$ is now an effective potential involving the cosmic time derivative of the scalar field.

It is instructive to view this equation in unitless form by defining $y = q/\eta$ and $x = r/r_m$ with $r_m = \sqrt{GM/a_0} = \sqrt{r_s l_M}$ ($r_s = 2GM/c^2$ is the Schwarzschild radius). Given that $V'(q) \approx 2q^2\dot{\phi}^2/\epsilon^2$ in this regime ($q \approx \eta$), that $a_0 = \sqrt{12}\eta|\dot{\phi}|/\epsilon$, and that $\nabla\phi = \eta GM/(c^2 r^2 q^2)$, then, in the case of spherical symmetry, equation (21) becomes

$$\eta^2 \frac{l_M}{r_s} \nabla^2 y = \frac{2}{x^4 y^3} - \frac{1}{12} y. \tag{22}$$

From this it is obvious that, in the limit of weak coupling, where $\eta^2 l_M/r_s \ll 1$, the relationship between q^2 and $\nabla\phi$ is effectively algebraic as in TeVeS (the theory approaches its AQUAL limit).

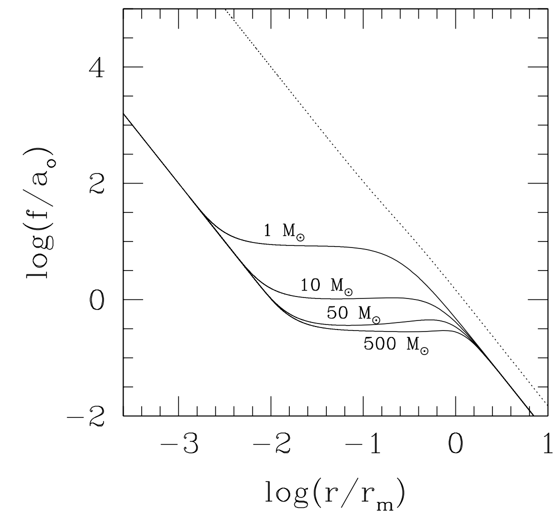

Figure 5. The log of the scalar force (f_s, solid curves) and the total force ($f_t = f_s + f_N$, dotted curve), in units of 10^{-8} cm s^{-2} plotted against the log of the radial distance from a point mass in units of the MOND radius $r_m = \sqrt{GM/a_0}$ for the biscalar theory. The various curves correspond to the indicated values of the source mass. The curves converge for $M > 50$ M$_\odot$ corresponding to the weak coupling limit (the AQUAL limit) of the theory.

In this case, the radial dependence of the scalar force is similar to that shown in Fig. 1 (dashed curves). This is consistent with galaxy rotation curves and would produce an anomalous acceleration in the Solar system similar to that of Fig. 4 (solid curve); that is, it satisfies all planetary constraints on deviations from $1/r^2$ attraction.

For a given value of η, the condition for weak coupling provides a lower limit on the source mass. Whenever

$$M > M_c \approx 6\eta^2 \frac{l_H c^2}{G} \approx 5 \times 10^{23} \eta^2\ \mathrm{M}_\odot, \tag{23}$$

then the weak coupling limit applies (here $l_H = c^2/H_0$ is the Hubble radius and I have taken $l_M = 6\ l_H$). If $\eta \approx 2 \times 10^{-12}$, this critical mass would correspond to a few solar masses. In other words, for larger mass, the weak coupling limit applies, and the form of the scalar force (as a function of r/r_m) is frozen as in TeVeS. But for smaller masses, the full differential equation (22) must be solved and the solution depends on the source mass (the presence of a critical mass in PCG was pointed out by Bekenstein 1988).

This is illustrated in Fig. 5 where we see the scalar force as a function of the scaled radius (r/r_m) for $\eta = 2 \times 10^{-12}$ and for various values of the source mass. The object is placed in the galaxy acceleration field near the position of the Sun where $q \approx \eta$ (the scalar force is comparable to the Newtonian force at large distance from the star). Here, if $M > 50$ M$_\odot$, the solution is fixed at the weak-coupling limit. For smaller values of the mass, the solution, and in particular the value of the plateau acceleration, depends on the source mass. If the source mass is 1 M$_\odot$, then the plateau acceleration is 8×10^{-8} cm s^{-2}. That is to say, unlike the weak coupling or AQUAL limit, the constant anomalous acceleration in the outer Solar system can be significantly larger than the MOND critical acceleration.

With $\eta = 2 \times 10^{-12}$ and $\epsilon^2 = 2 \times 10^4\ \eta^2$ (Sanders 2005), the resulting anomalous force (solid curve) is compared to the total force (dashed curve) in Fig. 6. Here, it is evident that, within 20 au the scalar force is precisely $1/r^2$ and 10^{-4} less than the Newtonian force. Moreover, the theory in this form produces a constant anomalous force that is consistent with the Pioneer acceleration but inconsistent with the reported limits, the variation of Kepler's constant out to

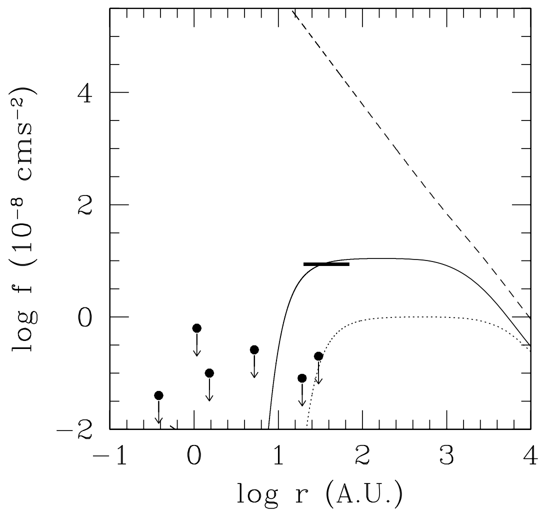

Figure 6. The dashed curve is the log of the total force ($f_t = f_s + f_N$), in units of 10^{-8} cm s^{-2} plotted against the log of the radial distance from the Sun in astronomical units for the biscalar theory. The solid curve is the anomalous force (the non-inverse-square component) for a scalar coupling $\eta = 2 \times 10^{-12}$ (with $\epsilon = 141\eta$) and the dotted curve is for $\eta = 0.9 \times 10^{-12}$ (with $\epsilon = 200\eta$). As in Fig. 4, the points show the planetary constraints on an anomalous non-inverse-square attraction.

Uranus and Neptune. Taking the parameters of the theory to be $\eta = 0.9 \times 10^{-12}$ and $\epsilon^2 = 4 \times 10^4\eta^2$ pushes the theory back to the AQUAL limit and produces the anomalous force shown as the dotted curve. This is consistent with all reported planetary constraints on inverse-square attraction and preferred frame effects as is TeVeS with the revised free function. In fact, it is identical in this respect to TeVeS with the revised free function (equation 8).

4.3 The Pioneer anomaly

TeVeS with the free function modified to be consistent with galaxy rotation curves (as in equation 8) predicts a constant anomalous acceleration beyond 100 au with magnitude $\approx 3 \times 10^{-9}$ cm s^{-2}. The same is true of BSTV in the limit of weak scalar coupling, $\eta < 10^{-12}$. Therefore, the theories may be constructed to be consistent both with galaxy rotation curves and with planetary constraints on $1/r^2$ attraction within the orbit of Neptune. In any case, an anomalous acceleration, a_p, in the outer Solar system is inevitable, provided that the scalar force is a monotonically decreasing function of radius. Consistency with galaxy rotation curves appears to require that $a_p \approx 0.3a_0$ as a lower limit. For BSTV, however, because the field determining the effective strength of the scalar coupling, q (or μ) is dynamical, it is possible that $a_p > a_0$ beyond 20 au while, in the outskirts of galaxies $a_p \approx 0.3a_0$. The same would probably be true of TeVeS with a dynamical μ. It is therefore tempting to identify the predicted constant acceleration with the Pioneer anomaly.

This is problematic because, as we see in Fig. 5, the Pioneer anomaly itself is inconsistent with reported limits on the variation of Kepler's constant out to Uranus and Neptune (Anderson et al. 1995). However, these stated limits may be overly stringent because they are based only on single spacecraft ranging measurements to these planets, and the formal uncertainties in the distances are almost certainly too optimistic. It should also be kept in mind that both Uranus and Neptune have not completed a single orbit period since the advent of precise astronomical positioning instrumentation and, therefore, their orbits are poorly known (Standish 2004, also private communication).

More recently, it has been claimed that such a large anomalous acceleration, if present beyond 20 au, would lead to secular and short-period signals in the orbits of the outer three planets – signals large enough to have been detected, given the present levels of accuracy (Iorio & Giudice 2006). The opposite conclusion has been reached by Page, Dixon & Wallin (2006) who proposed using distant asteroid orbits as a test for the Pioneer effect. It would seem fair to conclude that there is a lack of agreement about the nature of the gravitational field (as probed by planetary orbits) in the outer Solar system.

This is an important issue. If planetary motion beyond 20 au is inconsistent with the presence of the constant Pioneer acceleration, then the Pioneer anomaly is not due to a modification of gravity in the usual sense. If the planetary motion is consistent with the Pioneer anomaly, then it remains possible that this reported constant acceleration is due to the effect of a fifth force which becomes evident at low accelerations, as in relativistic theories of MOND.

5 CONCLUSIONS

Solar system phenomenology, in particular the tight limits on deviations from inverse-square attraction and the absence to high precision of local preferred frame effects, places strong constraints on multifield theories of modified dynamics as modified gravity. Specifically, any fifth force mediated by a scalar field must also be inverse square to high precision in the inner Solar system, at least within the orbit of Mars, but smaller than about 10^{-4} of the Newton–Einstein force to avoid producing observable preferred frame effects. I re-emphasize that this constraint on the magnitude of the scalar force is only an estimate based on heuristic arguments; a proper calculation of the preferred frame post-Newtonian parameters for TeVeS should be done. It is clear, however, that in these theories preferred frame effects, such as a polarization of the lunar orbit, should appear at some level.

In the Galaxy, at the position of the Sun, the Galactic gravitational acceleration is of the order of MOND acceleration $a_0 \approx 10^{-8}$ cm s^{-2}. In the context of multifield theories of MOND, this would imply that a fifth force acceleration is probably about as large as the Newtonian acceleration, or, in terms of scalar–tensor theory, $f_s/f_N \approx 1$. In other words, f_s/f_N must grow from 0.0001 within the orbit of Neptune to about 1 at a distance of 800 au where the Galactic gravitational acceleration becomes comparable to the solar attraction. Therefore, going outwards from the Sun to the Galactic environment, the scalar force must appear as an anomalous non-inverse-square acceleration (provided that the scalar force dependence on radius is monotonic). This effect, first noted for the stratified aquadratic theory (Sanders 1997), is an inevitable consequence of multifield theories and is evidenced by both TeVeS and BSTV.

In order to meet the Solar system constraints of precise inverse-square attraction and the absence of preferred frame effects, the scalar force must have the form demonstrated by the dashed curves in Fig. 1; that is, there must be a transition region between $1/r$ and $1/r^2$ attraction where the acceleration due to the scalar field is more or less constant. Therefore, not only must an anomalous acceleration appear in the outer Solar system (certainly beyond 100 au), but it must also be, to lowest order, constant with radius between 100 and 1000 au.

Both BSTV and TeVeS with the modified free function can provide this transition, but there is an important difference. In BSTV, it may be shown that for vanishing coupling strength ($\eta \rightarrow 0$) the Laplacian of q, the coupling strength field, may be neglected in the field equation for q, and the theory becomes, in effect, an aquadratic theory as in TeVeS. However, for finite η the form of q and hence the scalar force, in general, depends on the mass of the source and the coupling strength. For $\eta < 10^{-10}$, the AQUAL limit applies to galaxy scale masses, but the full differential equation must be solved for smaller masses. The practical consequence of this is that the plateau acceleration (where $f_s \leqslant a_0$) depends on the source mass. For a galaxy scale mass, this near-constant acceleration can be $\approx a_0/3$ (as required for rotation curves), but for a solar mass it may be near $10a_0$ if $\eta \approx 10^{-12}$. This possibility exists for TeVeS as well if the auxiliary field, μ, is given its own dynamics by writing a kinetic term proportional to $\mu_{,\alpha}\mu^{,\alpha}$ into the Lagrangian (this would provide a more familiar theory). Therefore, it is possible that the predicted constant anomalous force could be identified with the Pioneer anomaly.

But it is also evident from Figs 4 and 6 that no theory of MOND as modified gravity can satisfy the reported limits on deviation from inverse-square attraction at the orbits of Uranus and Neptune and be consistent with the Pioneer effect if the anomalous acceleration really does appear at radii as small as 20 au. This is because the Pioneer effect itself is inconsistent with these constraints. The constraints themselves are controversial. Basically, the solar gravitational field in the outer Solar system is not well understood, and this calls for a reconsideration of the orbits of the outer planets in the presence of a non-inverse-square acceleration.

It should also be noted that Milgrom (1994) had proposed a basis for MOND as modified inertia in which the particle action is a non-local functional of the entire particle trajectory. This is a completely different approach from the modified gravity theories discussed here, and could account for the possibility that the Pioneer spacecrafts on hyperbolic orbits feel the anomalous acceleration, but the planets on more circular orbits do not. For this reason, it would be of considerable interest to determine if the Pioneer anomaly first appears at the point where a spacecraft is boosted from a bound to an unbound orbit.

The significance of the Pioneer effect should not be understated. It may constitute the first evidence on a scale smaller than Galactic and extragalactic that there is more to gravity than we have supposed. The question of whether or not the Pioneer acceleration is a new physical effect and, if so, where the anomalous acceleration first appears requires re-analysis of the existing Pioneer data and, on the longer term, new space missions to confirm (or not) this important result (Turyshev, Nieto & Anderson 2004).

The possibility of such local tests is, in a sense, the 'holy grail' of modified gravity theories (as is the direct detection of new particles for dark matter theories). In this regard, Bekenstein & Majueigo (2006) had demonstrated that, in the context of TeVeS, MOND tidal stresses become anomalously large near saddle points in the local Solar system where the total weak-field force approaches zero, between the Earth and the Sun, for example. Space missions with sensitive accelerometers might detect such effects. This would indeed be a spectacular confirmation, but non-detection at the predicted level would not be a falsification of general modified gravity theories for MOND. In the biscalar theory, for example, the coupling strength field, also being dynamical, does not respond immediately to changes in the scalar force; once in the deep Newtonian regime, one cannot return to the MOND regime over relatively small distances. On the other hand, it does appear that any theory of MOND as modified gravity would require the presence of an anomalous acceleration in the outer Solar system (beyond 100 au) with a magnitude of at least a few tenths a_0.

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APPENDIX A: EDDINGTON–ROBERTSON POST-NEWTONIAN PARAMETERS IN THEORIES WITH DISFORMALLY RELATED METRICS

Let us suppose that, in the preferred frame, the relation between $\tilde{g}_{tt}$ and g_{tt} (the time–time components of the physical and Einstein metrics) can be written as

$$\tilde{g}_{tt} = \psi(\phi) g_{tt}, \tag{A1}$$

where ψ is a general function of ϕ, the scalar field. Further, take the Taylor expansion

$$\psi(\phi) = 1 + a\phi + \frac{1}{2}b\phi^2 + \cdots \tag{A2}$$

where $a = \psi'(0)$ and $b = \psi''(0)$. If the scalar field dynamics is described by the standard field Lagrangian, as it is for these theories in the inner Solar system (i.e. $L_s = \phi_{,\alpha}\,\phi^{,\alpha}$), then it is the case that

$$\phi = -a\frac{GM}{r}, \tag{A3}$$

as is usual in scalar–tensor theories. Taking

$$g_{tt} = -1 + h_{tt} \tag{A4}$$

where, to second order

$$h_{tt} = 2\frac{GM}{r} - 2\left(\frac{GM}{r}\right)^2. \tag{A5}$$

Then we find, to second order

$$\tilde{g}_{tt} = -1 + 2\left(1 + \frac{a^2}{2}\right)\frac{GM}{r} - 2\left(1 + a^2 + \frac{ba^2}{4}\right)\left(\frac{GM}{r}\right)^2. \tag{A6}$$

Redefining the mass $M' = [1 + (a^2/2)]M$, we then have

$$\tilde{g}_{tt} = 1 - 2\frac{GM'}{r} - 2\left(\frac{GM'}{r}\right)^2\left\{\frac{1 + a^2 + (a^2b/4)}{[1 + (a^2/2)]^2}\right\}. \tag{A7}$$

By identification with equation (14), we find

$$\beta = \frac{1 + a^2 + (a^2b/4)}{[1 + (a^2/2)]^2}. \tag{A8}$$

The condition for $\beta = 1$ is then $b = a^2$ or

$$\psi''(0) = \psi'(0)^2. \tag{A9}$$

This is obviously true if $\psi(\phi) = e^{a\phi}$ which would be the case for the particular transformation described by equation (1).

For a conformal transformation, it is also the case that $\tilde{g}_{rr} = \psi(\phi)g_{rr}$. Repeating the procedure above for $\tilde{g}_{rr}$ to first order, we find

$$\gamma = \frac{1 - a^2/2}{1 + a^2/2} \neq 1 \tag{A10}$$

[given that $a^2 = 2/(2\omega + 3)$, this returns the usual Brans–Dicke result].

Now suppose the transformation is disformal, and of a form which generalizes equation (1):

$$\tilde{g}_{\mu\nu} = u(\phi)g_{\mu\nu} - w(\phi)A_\mu A_n u, \tag{A11}$$

where $u(\phi)$ and $w(\phi)$ are unspecified functions of the scalar field. To post-Newtonian order, this is equivalent to multiplying the time–time and space–space components of $g_{\mu\nu}$ by different functions of the scalar field, for example, $\tilde{g}_{tt} = \psi(\phi)g_{tt}$ and $\tilde{g}_{rr} = \chi(\phi)g_{rr}$ where

$$\psi(\phi) = u(\phi) + w(\phi) \tag{A12}$$

and

$$\chi(\phi) = u(\phi). \tag{A13}$$

Taking

$$\chi(\phi) = 1 + a'\phi \tag{A14}$$

and repeating the above procedure for $\tilde{g}_{rr}$, we find

$$\gamma = \frac{1 - aa'/2}{1 + a^2/2}. \tag{A15}$$

Here $\gamma = 1$ if $a = a'$ or

$$\chi'(0) = -\psi'(0). \tag{A16}$$

Rewriting conditions (A9) and (A16) in terms of the functions u and w (via equations 12 and 13), we find that if

$$[u'(0) + w'(0)]^2 = u''(0) + w''(0) \tag{A17}$$

and

$$2u'(0) = -w'(0), \tag{A18}$$

then $\beta = \gamma = 1$. It is straightforward to confirm that the particular transformation given by equation (1) satisfies these two conditions.

It is important to note that Giannios (2005) had found two spherically symmetric static solutions for TeVeS: one in which $A^r = 0$ everywhere (as assumed here), and a second in which the vector field develops a non-zero radial component. In this second case, $\beta \neq 0$ and depends on the vector coupling strength parameter and the cosmological value of the scalar field. It is not yet determined if both solutions are stable.

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